# A scaling investigation for a Van der Pol circuit: normal form applied to a Hopf bifurcation

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**Abstract:** Scaling laws are generally associated with changes in the spatial structure of dynamical systems due to variations of control parameters. In the local bifurcation theory, when an equilibrium point changes stability from stable to unstable and a stable limit cycle shows up, we say the system has undergone a Hopf bifurcation. Some of the basic questions that remain to be explored about Hopf bifurcation are the regimes for which certain scaling laws exist and whether the exponents obtained for system obeying kinds of dynamics are valid for others. Based on this scenario, we explore the evolution towards the steady state at Hopf bifurcation in the Van der Pol circuit. The simplicity of Van der Pol circuit and the ability to generate a variety of behaviours motivate the choice of the system. Through the scaling analysis, we obtained the scaling properties and the critical exponents that characterise the bifurcation in study.

**Keywords:** Van der Pol oscillator; scaling properties; supercritical Hopf bifurcation; scaling formalism; critical exponents; steady state; normal form; local bifurcation theory; scaling law; class of universality.

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## 1 Introduction

Dynamical systems are generally dependent on control parameters and their variations. When a qualitative structure of the flow of solutions related to a dissipative dynamical system change due to the variation of the control parameters a bifurcation is observed. Bifurcation is the name given to the qualitative change in the flow of solutions of a fixed point[1] (Strogatz, 2015) due to a parameter variation. Different types of bifurcations are observed in a variety of systems including in dynamical population (Bashkirtseva and Ryashko, 2002; Strogatz, 2000), electric circuits (Georgiou and Romeo, 2015; Gardine et al., 2015), chemical reactions (Inarrea et al., 2015; Bakes et al., 2008), discrete mappings (Guckenheimer, 2008; Philominathan et al., 2011), laser (Virte et al., 2013; Doedel and Pando-L, 2011; Cavalcante and Leite-Rios, 2008) and many others (Guo and Luo, 2012; Luo and O'Connor, 2009; Yang et al., 2015).

When a variation of a control parameter produces a change of stability of a fixed point and, the topological modifications in the system can be confirmed by an investigation near the fixed point – therefore using a local analysis – a local bifurcation has happened. The scenario is different for a global bifurcation where invariant structures[2] collide with each other. This include also a collision between an invariant manifold and chaotic attractor yielding a destruction of the chaotic attractor. Consequently a major change in the topology of the system cannot be foreseen by a local analysis of fixed point. This class of bifurcation produces the so called crisis events (Grebogi et al., 1983a, 1983b; Leonel and McClintock, 2005, 2011).

In this paper, we discuss some scaling properties for the supercritical Hopf bifurcation observed in a Van der Pol circuit. It is known (Teixeira et al., 2015) that at a bifurcation the convergence to the steady state is described by a homogeneous function with a set of three critical exponents. They are used to describe the behaviour of short time, when the dynamics is described mostly by a constant plateau, for large enough time, that the dynamics converges to the steady state as a power law and for the crossover time, whereas the two regimes meet each other. For bifurcations in 1D mappings, it was considered particularly a family of logistic-like mappings (Teixeira et al., 2015). The convergence to the fixed point at a bifurcation was investigated considering both the phenomenological approach with the use of a scaling function and obtained at the end a scaling law with three critical exponents. Moreover, it was also considered an analytical procedure, transforming the equation of differences into a differential equation, which was easily integrated. The phenomenological investigation considered a set of three scaling hypotheses and specific plots to obtain the critical exponents for short time (exponent $\alpha$), large time (exponent $\beta$) and crossover time (exponent $z$). It was proved there (see Teixeira et al., 2015) the exponent $\alpha = 1$ is a constant while $\beta = -1/\gamma$ and $z = -\gamma$ do indeed depend on the nonlinearity of the mapping, defined as $\gamma$. The authors show for a logistic-like map of the type $x_{n+1} = Rx_n(1 - x_n^{\gamma})$, where $\gamma \geq 1$ the scaling law $z = \alpha/\beta$ is also observed. Soon after (Teixeira et al., 2015), the authors made a Taylor expansion of the second iterated of the mapping and described with success (see Leonel et al., 2015) the critical exponents for a period doubling bifurcation. For this bifurcation, the exponents do not depend on the nonlinearity of the mapping and are rather universal: $\alpha = 1$, $\beta = -1/2$ and $z = -2$. Near the bifurcation the dynamics is no longer described by a homogeneous function but for an exponential decay. The relaxation time of the exponential is described by a power law whose exponent $\delta = -1$ for three different bifurcations namely transcritical, pitchfork and period doubling.

An extension of the procedure was made also for a set of bifurcations observed in ordinary differential equations (see Leonel, 2016) considering a set of three important bifurcations of 1D flow namely: saddle-node, transcritical and supercritical pitchfork.

A question we want to answer here is what are the changes in the scaling properties when a stable fixed point loses stability giving rise to a one dimension higher attractor such as a limit cycle, particularly in a Van der Pol circuit? What would be the behaviour of critical exponents and the scaling law if a Hopf bifurcation characterises the dynamical system? In a recent paper (Silva and Leonel, 2018), the scaling properties were discussed for a Hopf bifurcation and, because after the bifurcation the attractor is a limit cycle in a plane, it turns out to be convenient to use polar coordinate to investigate the dynamics. The authors prove that instead of a single set of critical exponent, the dynamics needed two sets, among the exponent $\delta$. One set describes the convergence in the radial coordinate while the other set describes the evolution of the angular variable. The results obtained from the scaling analysis for the supercritical case of Hopf bifurcation are summarised in Table 1.

**Table 1** Table of critical exponents $\alpha_{\rho,\phi}$, $\beta_{\rho,\phi}$, $z_{\rho,\phi}$ and $\delta$ for a Hopf bifurcation

| | *Radial* ($\rho$) | *Angular* ($\phi$) |
|---|---|---|
| $\alpha$ | 1 | 1 |
| $\beta$ | $-1/2$ | 1 |
| $z$ | $-2$ | 1 |
| $\delta$ | $-1$ | – |

Despite the investigation of the convergence towards the steady state at these bifurcations leads us to the observables discussed, it is important to point out that a theoretical and experimental validation of them in dynamical systems, in general, has not been discussed yet, therefore justifying the present investigation. Based on this, our attempt here is to fill up this gap by discussing theoretically a specific system possible to be measured experimentally. Therefore, the main goal of this work is to obtain a theoretical verification of the scaling properties as well as the critical exponents that characterise the dynamics at the Hopf bifurcation exhibited by a form of the well-known Van der Pol circuit. In this paper, we examine the case where the resistance is replaced by a cubic nonlinearity. Other nonlinearities can also be considered, but we limit ourselves to cubic.

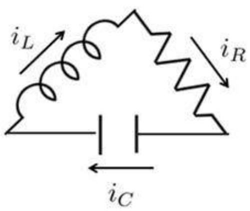
Nevertheless one may ask why we need to care about these scaling properties? Suppose one is doing an experimental investigation and the dynamical system studied exhibits a stable limit cycle oscillation and the set of equations that describes the dynamics are not known clearly. By examining the scaling properties for the dynamical system as well as the critical exponents involved, that characterise the dynamics near the bifurcation, this allows one to learn and classify the bifurcation and to extract information from the dynamics. In this way, models can be eliminated or supported depending on the scaling properties observed. The main findings in this work were obtained by considering the normal form of the equations describing the Van der Pol circuit, since it simplifies the form of the dynamics on the center manifold. The verification of the scaling properties characterising the dynamics at a Hopf bifurcation is achieved using the expression obtained from the normal form theory by considering its evolution towards the steady state. In this last, it is considered a direct solution of the differential equations, which are possible only in the normal form.

This paper is outlined as follows. The derivation of the nonlinear Van der Pol equations is made in Section 2. The computation of the normal form is presented in Section 3. Section 4 is devoted to making an analytical description of the convergence to the steady state at Hopf bifurcation leading to the critical exponents discussed in Silva and Leonel (2018). Discussions and conclusions are made in Section 5.

## 2 The Van der Pol equation

An example where nonlinear phenomena can be illustrated and studied is the Van der Pol circuit. Originally, the mathematical model developed for the circuit arose in connection to the nonlinear electronic circuits used in the first radios (Strogatz, 2015). The Van der pol circuit's applications are wide and comprehensive due to the ability to generate a large variety of nonlinear phenomena. As examples, one observe local bifurcations, period-doubling cascades, strange attractors, etc. (Strogatz, 2015; Buscarino et al., 2017). In general, Van der Pol circuit serves as base-model of self-excited oscillations in electronics, biology, neurology, physics and many others (Parlitz and Lauterborn, 1987; Cartwright and Littlewood, 2013; El-Abbasy, 1985; Levinson, 1949; Gollub et al., 1978; Holmes and Rand, 1978; Grasman et al., 1976, 1978; Grasman, 1978). However, in this paper, our goal is to focus in a new application yet not discussed for the Van der pol circuit, i.e., explore scaling properties observed through the convergence to the steady state at and nearby the Hopf bifurcation exhibited by this nonlinear system.

**Figure 1** Van der Pol circuit




In this section, we derive the equations and the eigenvalues that characterises the dynamical system in the present study. The Van der Pol circuit is a series $RLC$ circuit as shown in Figure 1. The relationship between the variables is easy to be obtained by combining the Kirchhoff's first and second laws. Thus, the Van der Pol circuit's equations in dimensionless units are given by

$$\begin{cases} \dot{x}_1 = -x_2 - g(x_1), \\ \dot{x}_2 = \qquad x_1. \end{cases} \tag{1}$$

In this paper we consider the case where $g(x_1)$ is a cubic function of the form, $g(x_1) = -\mu x_1 + x_1^3$, where $\mu \in \mathbb{R}$ controls the *negative* amount of resistance. For further developments, $\mu$ is treated as control parameter. The Van der Pol circuit posses a fixed point at $P_0 = (0, 0)$. Near the fixed point $P_0$ we have the Jacobian matrix

$$A = \begin{pmatrix} 0 & 1 \\ -1 & \mu \end{pmatrix}. \tag{2}$$

The eigenvalues of the matrix (2) considering the fixed point $P_0$ are

$$\lambda_{1,2} = \alpha(\mu) \pm i\beta(\mu), \tag{3}$$

where $\alpha(\mu) = \frac{\mu}{2}$ and $\beta(\mu) = \sqrt{1 - \alpha(\mu)^2}$. According to the linear stability theory, when $\mu < 0$ the fixed point turns a stable spiral whose sense of rotation depends on $\beta(\mu)$. For $\mu = 0$ the origin is still a stable spiral however the speed of convergence is different from $\mu < 0$. Finally, for $\mu > 0$ , there is an unstable spiral at the origin and, a stable and periodic limit cycle bifurcates from the fixed point.

In this section, we have derived the equations and the eigenvalues that characterise the Van der Pol circuit considering the fact this last exhibiting a supercritical Hopf bifurcation. In the next section the computation of the normal form is made properly.

## 3 Computation of the normal form

In the study of local bifurcation, the normal form theory is an advantageous approach, since it corresponds to the simpler analytical expression at which a dynamical system can be re-written through a convenient choice of a coordinate system without losing or changing the phase space topology of the originally. For a dynamical system whose dimension $N \geq 2$, the scaling analysis with normal form applied plays a major role, since the study of the scaling properties as well the derivation of the critical exponents through a simpler analytical expression is more attractive and easier to do than when one considers the original set of equations describing the dynamics.

To compute the normal form of the dynamical system (1) follows the theorem whose proof is given in appendix.

*Theorem 1:* Consider a two dimensional dynamical system described by the equations

$$\dot{x} = f(x, \mu),\ x \in \mathbb{R}^2,\ \mu \in \mathbb{R}, \tag{4}$$

where $f$ is a smooth function of its variables and time having for all sufficiently small $\mu$ the fixed point $x_0 = 0$ with eigenvalues

$$\lambda_{1,2} = \alpha(\mu) \pm i\beta(\mu). \tag{5}$$

If for $\mu = 0$ the following conditions are satisfied:

1 $\alpha(0) = 0,\ \beta(0) \neq 0$ $\quad (non - hyperbolicity)$

2 $\frac{d\alpha(\mu)}{d\mu}\Big|_{\mu=0} = 0$ $\quad (transversality)$

3 $l_1(0)^3 \neq 0$ $\quad (non - degeneracy)$ (6)

Then, the dynamical system expressed in (4) can be rewritten in the following normal form

$$\begin{cases} \dot{y}_1 = \alpha(\mu)y_1 - \beta(\mu)y_2 + (ay_1 - by_2)((y_1)^2 + (y_2)^2), \\ \dot{y}_2 = \alpha(\mu)y_2 + \beta(\mu)y_1 + (ay_2 + by_1)((y_1)^2 + (y_2)^2). \end{cases} \tag{7}$$

The theorem says for a generic dynamical system in two dimensions, if it is proved that it satisfies the conditions of the Hopf bifurcation theorem, then its normal form is known to be described by (7). From the Theorem 1 follows its corollary:

*Corrolary 2:* Suppose the two-dimensional system

$$\dot{x} = f(x, \mu), \; x \in \mathbb{R}^2, \; \mu \in \mathbb{R}, \tag{8}$$

and, $x_0$ its fixed point. At $\mu = 0$, the dynamical system (8) is said to exhibit a Hopf bifurcation. Suppose further that for $\mu < \mu_0\,(\mu > \mu_0)$, (8) has a pair of complex-conjugate eigenvalues with positive real part and, for $\mu > \mu_0\,(\mu < \mu_0)$, (8) has a pair of complex-conjugate eigenvalues with negative real part. Then,

1 For $l_1 < 0$, which happens when $a < 0$, the fixed point $x_0$ is said to be asymptotically stable at $\mu = \mu_0$. Although, at $\mu > \mu_0\,(\mu < \mu_0)$ an unique stable (unstable) curve of periodic solutions bifurcates from the unstable (stable) fixed point. In this case, the dynamical system exhibits the so called supercritical Hopf bifurcation.

2 For $l_1 > 0$, which happens when $a > 0$, the fixed point $x_0$ is said to be unstable at $\mu = \mu_0$. However, at $\mu < \mu_0\,(\mu > \mu_0)$ an unique unstable (stable) curve of periodic solutions bifurcates from the stable (unstable) fixed point. In this case, the dynamical system exhibits the so called subcritical Hopf bifurcation.

3 For $l_1 = 0$, nothing can be said about the dynamics of (8).

The corollary above establishes whether the dynamical system in study exhibits the supercritical or the sub-critical cases of the Hopf bifurcation.

Backing to the Van der pol circuit, in Section 2 it was shown the eigenvalues of the Jacobian matrix are $\alpha(\mu) \pm i\beta(\mu) = \frac{\mu}{2} \pm i\frac{\sqrt{4-\mu^2}}{2}$. Note that $\alpha(0) = 0$ and $\beta(0) = -1 \neq 0$. Also, $\frac{d\alpha(\mu)}{d\mu}\big|_{\mu=0} = \frac{1}{2} \neq 0$. Finally, $l_1(0) = -1/8 \neq 0$. Therefore, all the conditions of the Theorem 1 are satisfied. Hence, the normal form of the dynamical system that characterises Van der Pol circuit is given by

$$\begin{cases} \dot{y}_1 = \alpha(\mu)y_1 - \beta(\mu)y_2 + (ay_1 - by_2)((y_1)^2 + (y_2)^2), \\ \dot{y}_2 = \alpha(\mu)y_2 + \beta(\mu)y_1 + (ay_2 + by_1)((y_1)^2 + (y_2)^2). \end{cases}$$

where $a$ and $b$ are constants.

In this section, the normal form of Van der Pol's equations is obtained. The theorem proposed acts as a shortcut to obtain the normal form for dynamical systems characterised by occurrence of the Hopf bifurcation.

## 4 Convergence to the steady state and scaling properties

Our main goal in this section is to investigate the scaling properties observed in the convergence to steady state at the Hopf bifurcation that characterises the Van der Pol circuit. Before the bifurcation, the dynamics converges to a fixed point which is asymptotic stable. However, at the bifurcation point, the fixed point point loses stability and after the bifurcation it repeals the dynamics which converge to a closed orbit in a plane, i.e., a limit cycle of period 1. Because the attractor is closed cycle in a plane, the use of polar coordinates is the better approach to describe the dynamics and, hence investigate the scaling properties as well the critical exponents for the Hopf bifurcation.

From the knowledge of $\beta$ and $l_1(0)$, the set of equations (7) can be written in polar coordinates as

$$\begin{cases} \dot{\rho} = \alpha\rho - \rho^3 \\ \dot{\phi} = \beta + b\rho^2, \end{cases} \tag{9}$$

where $\rho$ and $\phi$ describe the radial and the angular coordinates, respectively. Besides that, $\alpha$ controls the stability of the fixed point, $\beta$ gives the frequency of infinitesimal oscillations and $b$ is a free parameter.

We start considering the evolution towards the fixed point at the bifurcation point, i.e., at $\alpha = 0$. The differential equation is written as

$$\frac{d\rho}{dt} = -\rho^3. \tag{10}$$

A straightforward integration gives

$$\rho(t) = \frac{\rho_0}{\sqrt{1 + 2t\rho_0^2}}. \tag{11}$$

Let us now discuss the implications of equation (11) for specific ranges of $t$. For sufficiently short time we realise that $\rho(t) \propto \rho_0$. Since this is a constant for short time, the exponent $\alpha_\rho$ obtained from the hypothesis $\rho(t) \propto \rho_0^{\alpha_\rho}$, we end up with the conclusion of $\alpha_\rho = 1$. However, for sufficiently long times we have

$$\rho(t) \propto t^{-1/2}. \tag{12}$$

The hypothesis for long time is that $\rho(t) \propto t^{\beta_\rho}$, therefore we conclude that $\beta_\rho = -1/2$. From the scaling law $z_\rho = \alpha_\rho/\beta_\rho$ we obtain that $z_\rho = -2$.

We then discuss the case $\alpha \neq 0$ considering the convergence to the steady state at the neighbourhood of a Hopf bifurcation. Near a bifurcation the convergence to the steady state is described by an empirical function of the type $\rho(t) - \rho^* = (\rho_0 - \rho^*)e^{-t/\tau}$ where $\rho^*$ is the value of $\rho$ at the equilibrium, $\rho_0$ is the initial condition for $\rho$ at $t = 0$, $\tau \propto \alpha^\delta$ where $\tau$ is the relaxation time with $\alpha$ denoting the distance from the bifurcation measured in the control parameter and $\delta$ is the critical exponents driving the speed of convergence to the steady state. We have then to solve the following differential equation $\frac{d\rho}{dt} = \alpha\rho - \rho^3$. A direct integration gives

$$\rho(t) - \sqrt{\alpha} \simeq \frac{\sqrt{\alpha}}{2}e^{-2\alpha t}. \tag{13}$$

This result furnishes the relaxation exponent $\delta = -1$. The results obtained in this section by considering an analytical approach are in complete agreement with those shown in Table 1 for variable $\rho$.

A next step is to investigate the angular equation. We consider first the case of $\alpha = 0$. The differential equation, when incorporated the solution of $\rho(t)$, is written as $\frac{d\phi}{dt} = \beta + b\frac{\rho_0^2}{1+2t\rho_0^2}$. After integration we obtain the following

$$\phi(t) = \phi_0 + \beta t + \frac{b}{2}\ln(1 + 2t\rho_0^2). \tag{14}$$

Let us now discuss the implications of equation (14) for specific ranges of $t$. Considering the case where $\beta t + \frac{b}{2}\ln(1 + 2t\rho_0^2) \ll \phi_0$, we realise that $\phi(t) \propto \phi_0$ leading to $\alpha_\phi = 1$. However, in the case $\beta t \gg \phi_0 + \frac{b}{2}\ln(1 + 2t\rho_0^2)$, we obtain $\phi(t) \propto t$ giving $\beta_\phi = 1$. The last case is obtained when $\beta t = \phi_0 + \frac{b}{2}\ln(1 + 2t\rho_0^2) \cong \phi_0$, which gives $z_\phi = 1$. Again the results obtained for variable $\phi$ are in complete agreement with the results show in Table 1.

## 5 Conclusions

We derived a normal form to investigate the scaling properties as well as the critical exponents that drives the dynamics at and near at a Hopf bifurcation for the Van der Pol circuit. The assumption made from the problem was the fact that the scaling properties studied in Leonel et al. (2015), Leonel (2016), Silva and Leonel (2018) and Teixeira et al. (2015) have not been applied yet theoretically for the Van der Pol circuit. We proved the normal form of the circuit at the bifurcation can be written as

$$\begin{cases} \dot{y}_1 = \alpha y_1 - \beta y_2 + (ay_1 - by_2)((y_1)^2 + (y_2)^2), \\ \dot{y}_2 = \alpha y_2 + \beta y_1 + (ay_2 + by_1)((y_1)^2 + (y_2)^2). \end{cases}$$

Since after the bifurcation the dynamics live in the plane, the use of polar coordinates played a major role to describe the evolution towards the steady-state, once the supercritical Hopf bifurcation is indeed characterised by an evolution of attractors (Silva and Leonel, 2018).

It is known (Teixeira et al., 2015) that, at a bifurcation, the evolution to the steady state is described by a homogeneous function. Three exponents namely, $\alpha$ (short time), $\beta$ (large time) and $z$ (crossover time), describe the behaviour of both short and large time dynamics. For short time, a constant plateau of the distance of the dynamical variable to the fixed point against the time is observed. Soon after the dynamics heads towards the asymptotic state, a power law in time is observed. The crossover from one regime to the other one depends on the initial distance from the steady state. For one-dimensional mappings, the most generic case can be considered as the logistic-like map, written as $x_{n+1} = Rx_n(1 - x_n^\gamma)$ where $x$ is the dynamical variable, $R$ is a control parameter and $n$ denotes the discrete time. The relevant scaling law relating the three exponents is $z = \alpha/\beta$. Near the bifurcation the decay is mostly described by an exponential function, whose relaxation time depends on how far is the parameter from the bifurcation parameter. A fourth exponent $\delta$ characterises such a dynamics.

The formalism is not applied only for mappings but can also be used in ordinary differential equations (see Leonel, 2016). Three local bifurcations of 1D flow where investigated namely saddle-node, transcritical and supercritical pitchfork. Although, the bifurcations discussed in the previous works were especially characterised by the matter fact that the stability of the fixed points changes as the system's parameters was varied. However, one may ask about would happen with the scale properties of the system if the stability of the fixed point were switched off from stable to unstable and a periodic solution showed up, i.e., what would be the behaviour of critical exponents and the scaling law if a Hopf bifurcation characterises the dynamical system?

In the present paper, our results extend the formalism discussed in the previous works to be used in dynamical system of dimension $N \geq 2$ subjected to the supercritical Hopf bifurcation. It is important to emphasise, to the best of our knowledge, that a theoretical and experimental validation of exponents have not yet been considered in such type of system. Based on this, our intent in this paper was to fill up this gap discussing theoretically the Van der pol circuit.

Scaling laws' relevance is wide and comprehensive. As an example, suppose an experimental scientist studying a dynamical system that exhibits a stable limit cycle oscillation and the set of equations are not known. By examining the dynamical system's scaling properties as well as the critical exponents that characterise the dynamics near of some bifurcation, this allows to derive critical information about the system and characterise, drop or support models depending on the scaling properties observed.

Hopf Bifurcations become extremely important phenomena in experimental systems mainly for those involving the laser threshold. In this direction, the experimental verification of the scaling formalism developed helps in offering an effective method for its detection and, hence a way of preventing or informing about crises and catastrophes, which can have critical dynamical consequences (Duarte et al., 2012; Schuster, 1996; Rogister et al., 2003).

Given these points, our study identifies that the scaling properties at and near at the Hopf bifurcation as well as the scaling properties for others bifurcations provide an alternative form to investigate and classify different types of bifurcation. Since we theoretically validated the scaling properties for the Hopf bifurcation in the Van der Pol circuit, new rigorous experimental research of such system are needed to confirm our hypotheses.

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## Notes

1 Eventually the terminology is also used to periodic orbits.

2 As invariant structures we want to say invariant manifold, either stable, heading towards the saddle point, or unstable, moving away from the saddle point.

3 $l_1$ is the first Lyapunov coefficient whose expression is given by

$$\begin{aligned} l_1(0) = & \frac{1}{16}(f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy}) \\ & + \frac{1}{16\beta(0)}(f_{xy}(f_{xx} + f_{yy}) \\ & - g_{xy}(g_{xx} + g_{yy}) \\ & - f_{xx}g_{xx} + f_{yy}g_{yy}) \end{aligned}$$

with $f_{xy} = \frac{\partial^2 f_\mu}{\partial_x \partial_y} \Big|_{\mu=0}(0,0)$ and so on.

## Appendix

### *Proof of Theorem 1*

*Proof:* Consider a dynamical system

$$\dot{x} = f(x, \mu), \ x \in \mathbb{R}^2, \ \mu \in \mathbb{R}. \tag{15}$$

Assuming $f$ smooth, it has for $((0,0),0)$ the fixed point $x_0 = 0$ with eigenvalues $\lambda_{1,2} = \pm i\beta(0), \ \beta(0) \neq 0$.

*Proof:* Assuming the two-dimensional system (15) above, the Implicit function theorem establishes there is a sufficient small $\epsilon$ such that

$$x : (-\epsilon, \epsilon) \to \mathbb{R}^2$$

$$\mu \mapsto x(\mu)$$

describes a smooth function and $f(x(\mu), \mu) = 0 \ \ \forall \mu \in (-\epsilon, \epsilon)$. Assuming without loss of generality that $x_0 = 0$ is the fixed point of the dynamical system and $|\mu|$ is sufficiently small, then

$$f = f(x, \mu) = f(0, \mu) + \frac{df}{dx}(0, \mu)x + \frac{1}{2!}\frac{d^2 f}{dx^2}(0, \mu)x^2 + \ldots$$

Although, as long as $f(0, \mu) = 0$, this implies the system (15) may be reduced to the following expression

$$f(x, \mu) = A(\mu)x + F(x, \mu), \tag{16}$$

where $A(\mu)$ is the Jacobian matrix and $F(x, \mu)$ is a smooth function whose terms have Taylor expansions in $x$ starting with, at least, quadratic terms. The Jacobian matrix $A(\mu)$ can be written as

$$A = \begin{pmatrix} a(\mu) \ b(\mu) \\ c(\mu) \ d(\mu) \end{pmatrix},$$

with smooth functions of $\mu$ as its elements. The eigenvalues associated with the Jacobian matrix are determined by the characteristic polynomial expressed by

$$\lambda^2 - \lambda(a(\mu) + d(\mu)) + a(\mu)d(\mu) - c(\mu)b(\mu) = 0. \tag{17}$$

Establishing $(a(\mu) + d(\mu))$ as $\sigma(\mu)$ and $(a(\mu)d(\mu) - c(\mu)b(\mu))$ as $\delta(\mu)$ the eigenvalues are

$$\lambda_{1,2} = \frac{\sigma(\mu) \pm \sqrt{\sigma^2(\mu) - 4\delta(\mu)}}{2}. \tag{18}$$

However, for sufficient small $|\mu|$, we have

$$\lambda_1 = \lambda(\mu) = \alpha(\mu) + i\beta(\mu), \tag{19}$$

and

$$\lambda_2 = \overline{\lambda(\mu)} = \alpha(\mu) - i\beta(\mu).$$

where $\alpha(\mu) = \frac{1}{2}\sigma(\mu)$ and $\beta(\mu) = \frac{1}{2}\sqrt{4\delta(\mu) - \sigma^2(\mu)}$. Considering $\alpha(0) = 0$ and $\beta(0) \neq 0$ the eigenvalues become

$$\lambda_{1,2} = \pm i\beta(0), \tag{20}$$

From the knowledge of the eigenvalues, follows the Hopf bifurcation theorem (Kuznetsov, 1998; Marsden and McCracken, 1976).

*Theorem 3:* Suppose a two-dimensional system

$$\dot{x} = f(x, \mu), \ x \in \mathbb{R}^2, \ \mu \in \mathbb{R}, \tag{21}$$

with smooth $f$ has, for all sufficiently small $\mu$, the fixed point $x_0 = 0$ with eigenvalues

$$\lambda_{1,2} = \alpha(\mu) \pm i\beta(\mu), \tag{22}$$

where $\alpha(0) = 0$, $\beta(0) \neq 0$.

Let the following conditions be satisfied:

1 $\alpha^{'}(0) \neq 0$.

2 $l_1(0) \neq 0$, where $l_1$ is the first Lyapunov coefficient.

Then, the system (21) exhibits a Hopf Bifurcation.

*Remark 1:* The Theorem 3, whenever its conditions are satisfied, guarantees that a Hope bifurcation happens for the dynamical system considered.
Once the main interest is determining the normal form for the dynamical systems with this particular kind of bifurcation, we must apply the normal form theory at first. Although it is necessary to write the Jacobian matrix $A(\alpha)$ in the canonical real (Jordan) form first.

*Lemma 4:* Let $A(\mu)$ be a squared matrix with smooth functions of $\mu$ as its elements. Assume further that $A(\mu)$ has complex-conjugate eigenvalues. The canonical real (Jordan) form of $A(\mu)$ is written as

$$\mathbf{J} = \begin{pmatrix} \alpha(\mu) & -\beta(\mu) \\ \beta(\mu) & \alpha(\mu) \end{pmatrix}. \tag{23}$$

*Proof:* Let the Jacobian matrix $A(\mu)$ be written as

$$A = \begin{pmatrix} a(\mu) b(\mu) \\ c(\mu) d(\mu) \end{pmatrix},$$

with smooth functions of $\mu$ as its elements. The eigenvalues associated with the Jacobian matrix are determined by the characteristic polynomial expressed by

$$P(\lambda, \mu) = \lambda^2 - \lambda(a(\mu) + d(\mu)) + a(\mu)d(\mu) - c(\mu)b(\mu) = 0, \tag{24}$$

leading to

$$\lambda_{1,2} = (a(\mu) + d(\mu)) \pm \sqrt{(a(\mu) + d(\mu))^2 - 4(a(\mu)d(\mu) - c(\mu)b(\mu))}. \tag{25}$$

Assuming the discriminant of the polynomial $P$ is negative, we will have two complex conjugate eigenvalues

$$\lambda = \alpha + i\beta \ , \ \overline{\lambda} = \alpha - i\beta.$$

Let $w = u + iv$ be the complex eigenvector of $A(\mu)$. The eigenvalue equation gives

$$Aw = \lambda w,$$
$$A(u + iv) = (\alpha(\mu) + i\beta(\mu))(u + iv),$$
$$Au + iAv = (\alpha(\mu)u - \beta(\mu)v) + i(\beta(\mu)u + \alpha(\mu)v).$$

Therefore, the canonical real (Jordan) form of matrix $A(\mu)$ is given by

$$\mathbf{J} = \begin{pmatrix} \alpha(\mu) & -\beta(\mu) \\ \beta(\mu) & \alpha(\mu) \end{pmatrix}. \tag{26}$$

Once the canonical real (Jordan) form of the matrix $A(\mu)$ is obtained, we have now to dedicate our effort to determine the normal form of the dynamical system exhibiting the Hopf bifurcation. Here the process to obtain the normal form consists of two steps:

1 write a simplified expression for the Jordan matrix

2 apply the normal form theory properly.

## *Step 1: A simplification for the Jordan matrix J*

Here, we propose an alternative expression for the Jordan matrix **J** that consists of decomposing it into a sum of a symmetric and skew-symmetric matrix through the action of the operator residue, $\hat{\mathbb{V}}$, on **J**. This decomposition will simplify the computation of the normal form searched.

*Lemma 5:* Let **J** be the canonical real (Jordan) matrix of a dynamical system that exhibits the Hopf bifurcation. The action of the operator residue, $\hat{\mathbb{V}}$, on **J** leads to

$$\hat{\mathbb{V}}\mathbf{J} = \begin{pmatrix} \alpha(\mu) & 0 \\ 0 & \alpha(\mu) \end{pmatrix} + \begin{pmatrix} 0 & -\beta(\mu) \\ \beta(\mu) & 0 \end{pmatrix}, \tag{27}$$

where

$$\tilde{\alpha} = \begin{pmatrix} \alpha(\mu) & 0 \\ 0 & \alpha(\mu) \end{pmatrix} \tag{28}$$

is the $\tilde{\alpha}$-reduced matrix and

$$\tilde{J} = \begin{pmatrix} 0 & -\beta(\mu) \\ \beta(\mu) & 0 \end{pmatrix}. \tag{29}$$

is $\tilde{J}$-reduced matrix.

*Proof:* The action of operator residue on **J** is expressed by

$$\hat{\mathbb{V}}\mathbf{J} = \frac{1}{2}(\mathbf{J} + \mathbf{J}^T) + \frac{1}{2}(\mathbf{J} - \mathbf{J}^T).$$

This gives that

$$\hat{\mathbb{V}}\mathbf{J} = \frac{1}{2}\begin{pmatrix} 2\alpha(\mu) & 0 \\ 0 & 2\alpha(\mu) \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & -2\beta(\mu) \\ 2\beta(\mu) & 0 \end{pmatrix}.$$

Therefore,

$$\hat{\mathbb{V}}\mathbf{J} = \begin{pmatrix} \alpha(\mu) & 0 \\ 0 & \alpha(\mu) \end{pmatrix} + \begin{pmatrix} 0 & -\beta(\mu) \\ \beta(\mu) & 0 \end{pmatrix}. \tag{30}$$

*Step 2: Computation of the normal form*

From the knowledge of $\tilde{\alpha}$ and $\tilde{J}$, follows the normal form theorem.

*Theorem 6:* Let $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ be $C^r$ system of differential equations with $f(0) = 0$ and $Df(0)\mathbf{x} = \mathbf{J}$. Choose a complement $G^k$ for $\mathcal{L}(H^{(k)})$ in $(H^k)$, so that

$$\mathcal{L}(H^{(k)}) \oplus G^{(k)} = H^{(k)}. \tag{31}$$

Then, there is an analytical change of coordinates in the neighbourhood of the origin which transforms the system $\dot{\mathbf{x}} = f(\mathbf{x})$ into $\dot{\mathbf{y}} = \mathbf{g}(\mathbf{y}) = g^{(1)}(\mathbf{y}) + g^{(2)}(\mathbf{y}) + \ \ldots \ + g^{(r)}(\mathbf{y}) + R_r$ with $\mathbf{J}(\mathbf{y}) = g^{(1)}(\mathbf{y})$ and $R_r = O(|\mathbf{y}|^r)$.

The detailed proof of this Theorem 6 is found in Marsden and McCracken (1976).

*Remark 2:* The normal form theory is a powerful approach to construct a simplified analytical expression for a nonlinear dynamical system. It is based upon the computation of nonlinear transformations which reduce the coupling among the terms in the original dynamical system. Although, here the computation is done by applying the approach on $\tilde{\alpha}$ and $\tilde{J}$ matrices. In the computation of the normal form, the first-order normal form problem consists of looking at for $g^{(1)}(\mathbf{y})$. However, according to Theorem 6 $g^{(1)}(\mathbf{y}) = \mathbf{J}(\mathbf{y})$. Based on this, our work will be only limited to determine $g^{(2)}(\mathbf{y})$ and, $g^{(3)}(\mathbf{y})$.

*Lemma 7:* The $\tilde{\alpha}$-reduced matrix does not bring contributions to the normal form of dynamical systems that exhibit the Hopf bifurcation.

*Proof:* For the second-order normal form problem, the basis for $H^{(2)}$ consists of 6 monomials

$$\begin{bmatrix} (y_1)^2 \\ 0 \end{bmatrix}, \begin{bmatrix} y_1\, y_2 \\ 0 \end{bmatrix}, \begin{bmatrix} (y_2)^2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ (y_1)^2 \end{bmatrix}, \begin{bmatrix} 0 \\ y_1\, y_2 \end{bmatrix}, \begin{bmatrix} 0 \\ (y_2)^2 \end{bmatrix}.$$

The monomials are represented respectively by $\vec{m}_1, \vec{m}_2, \ldots, \vec{m}_6$. In this case, the functions $\vec{g}^{(2)}$ and $\vec{h}^{(2)}$ are elements from the space of $H^{(2)}$.

Although to find the contribution of the $\tilde{\alpha}$-reduced matrix to the normal form, it is necessary to obtain first the complementary sub-space $G^{(2)}$. At the space of $H^{(2)}$, whose basis is composed by the six monomials $\vec{m}_j$, $\mathcal{L}_{\tilde{\alpha}}(H^{(2)})$ can be represented by

$$\mathcal{L}_{\tilde{\alpha}}(H^{(2)}) = \begin{bmatrix} \alpha(\mu) & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha(\mu) & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha(\mu) & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha(\mu) & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha(\mu) & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha(\mu) \end{bmatrix}.$$

However, note the determinant of $\mathcal{L}(H^{(2)})$ is non-zero. According to the theory, when the determinant of $\mathcal{L}(H^{(k)})$ admits non-zero values, this automatically implies the $k$-order terms can be eliminated from the normal form. Once the normal form does not have second-order terms, we repeat the process for the third-order. In this new case, the basis of $H^{(3)}$ is now composed by eight monomials

$$\begin{bmatrix} (y_1)^3 \\ 0 \end{bmatrix}, \begin{bmatrix} (y_1)^2\, y_2 \\ 0 \end{bmatrix}, \begin{bmatrix} y_1\, (y_2)^2 \\ 0 \end{bmatrix}, \begin{bmatrix} (y_2)^3 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 \\ (y_1)^3 \end{bmatrix}, \begin{bmatrix} 0 \\ (y_1)^2\, y_2 \end{bmatrix}, \begin{bmatrix} 0 \\ y_1\, (y_2)^2 \end{bmatrix}, \begin{bmatrix} 0 \\ (y_2)^3 \end{bmatrix}.$$

Therefore, the $\mathcal{L}_{\tilde{\alpha}}(H^{(3)})$ is given by

$$\mathcal{L}_{\tilde{\alpha}}(H^{(3)}) = \begin{bmatrix} 2\alpha(\mu) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2\alpha(\mu) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2\alpha(\mu) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\alpha(\mu) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\alpha(\mu) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\alpha(\mu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2\alpha(\mu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\alpha(\mu) \end{bmatrix}.$$

We notice again the determinant of $\mathcal{L}(H^{(3)})$ is also non-zero. So the third-order terms can be eliminated from the normal form. Hence, the $\tilde{\alpha}$-reduced matrix does not bring contributions to the normal form of dynamical systems that exhibit the Hopf bifurcation.

*Lemma 8:* Only the $\tilde{J}$-reduced matrix bring contributions to the normal form of dynamical systems that exhibit the Hopf bifurcation.

*Proof:* Let us now consider the $\tilde{J}$-reduced matrix. For the second-order normal form problem, the basis for $H^{(2)}$ consists of six monomials of the type

$$\begin{bmatrix} (y_1)^2 \\ 0 \end{bmatrix}, \begin{bmatrix} y_1\, y_2 \\ 0 \end{bmatrix}, \begin{bmatrix} (y_2)^2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ (y_1)^2 \end{bmatrix}, \begin{bmatrix} 0 \\ y_1\, y_2 \end{bmatrix}, \begin{bmatrix} 0 \\ (y_2)^2 \end{bmatrix}.$$

Again, in order to determine the contributions of $\tilde{J}$-reduced matrix for normal form, it is necessary to obtain first the complementary sub-space $G^{(2)}$. At the $H^{(2)}$ space, whose basis is composed by the six monomials $\vec{m}_j$, $\mathcal{L}_{\tilde{J}}(H^{(2)})$ can be represented by the following matrix

$$\mathcal{L}_{\tilde{J}}(H^{(2)}) = \begin{bmatrix} 0 & \beta & 0 & \beta & 0 & 0 \\ -2\beta & 0 & 2\beta & 0 & \beta & 0 \\ 0 & -\beta & 0 & 0 & 0 & \beta \\ -\beta & 0 & 0 & 0 & \beta & 0 \\ 0 & -\beta & 0 & -2\beta & 0 & 2\beta \\ 0 & 0 & -\beta & 0 & -\beta & 0 \end{bmatrix}.$$

However, note the determinant of $\mathcal{L}(H^{(2)})$ is non-zero. According to the theory, when the determinant of $\mathcal{L}(H^{(2)})$ admits non-zero values, this automatically implies the second-order terms can be eliminated from the normal form. Again the normal form does not have second-order terms, we repeat the process for the third-order. For this new case, the basis of $H^{(3)}$ is then composed by eight monomials

$$\begin{bmatrix} (y_1)^3 \\ 0 \end{bmatrix}, \begin{bmatrix} (y_1)^2\, y_2 \\ 0 \end{bmatrix}, \begin{bmatrix} y_1\, (y_2)^2 \\ 0 \end{bmatrix}, \begin{bmatrix} (y_2)^3 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 \\ (y_1)^3 \end{bmatrix}, \begin{bmatrix} 0 \\ (y_1)^2\, y_2 \end{bmatrix}, \begin{bmatrix} 0 \\ y_1\, (y_2)^2 \end{bmatrix}, \begin{bmatrix} 0 \\ (y_2)^3 \end{bmatrix}.$$

Therefore $\mathcal{L}_{\tilde{J}}(H^{(3)})$ is given by

$$\mathcal{L}_{\tilde{J}}(H^{(3)}) = \begin{bmatrix} -\beta & 0 & 0 & 0 & 0 & \beta & 0 & 0 \\ 0 & -\beta & 0 & 0 & -3\beta & 0 & 2\beta & 0 \\ 0 & 0 & -\beta & 0 & 0 & -2\beta & 0 & 3\beta \\ 0 & 0 & 0 & -\beta & 0 & 0 & -\beta & 0 \\ 0 & 0 & 0 & 0 & 6\beta & 0 & -6\beta & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\beta & 0 & -2\beta \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Note the $det(\mathcal{L}_{\tilde{J}}(H^{(3)}))$ is zero. Hence the third-order terms cannot be eliminated. The only contributions for the normal form expression come from the $\tilde{J}$-reduced matrix and are of the third-order.

Although, as long as $det(\mathcal{L}_{\tilde{J}}(H^{(3)}))$ is zero, then there is a complementary sub-space $G^{(3)}$ to be determined. Once $(\mathcal{L}_{\tilde{J}}(H^{(3)})$ is rank 6 and the rank of $(H^{(3)})$ is 8, the null-space $G^{(3)}$ must be given by a linear combination of two vectors that can be represented as

$$g^{(3)} = a \begin{pmatrix} y_1^3 + y_1 y_2^2 \\ y_1^2 y_2 + y_2^3 \end{pmatrix} + b \begin{pmatrix} -y_1^2 y_2 - y_2^3 \\ y_1^3 + y_1 y_2^2 \end{pmatrix}, \tag{32}$$

where $a$ and $b$ are constants.

According to Theorem 6, the normal form of a two-dimensional system that satisfies the conditions from the theorem of Hopf bifurcation is given by

$$\begin{cases} \dot{y}_1 = \alpha(\mu) y_1 - \beta(\mu) y_2 + (a y_1 - b y_2)((y_1)^2 + (y_2)^2), \\ \dot{y}_2 = \alpha(\mu) y_2 + \beta(\mu) y_1 + (a y_2 + b y_1)((y_1)^2 + (y_2)^2). \end{cases} \tag{33}$$

This result ends the proof of Theorem 1.