ZDM Mathematics Education (2014) 46:507–515
DOI 10.1007/s11858-014-0615-x

SURVEY PAPER

Research on calculus: what do we know and where do we need to go?

Chris Rasmussen · Karen Marrongelle · Marcelo C. Borba

Accepted: 10 July 2014 / Published online: 30 July 2014

Abstract In this introductory paper we take partial stock of the current state of field on calculus research, exemplifying both the promise of research advances as well as the limitations. We identify four trends in the calculus research literature, starting with identifying misconceptions to investigations of the processes by which students learn particular concepts, evolving into classroom studies, and, more recently research on teacher knowledge, beliefs, and practices. These trends are related to a model for the cycle of research and development aimed at improving learning and teaching. We then make use of these four trends and the model for the cycle of research and development to highlight the contributions of the papers in this issue. We conclude with some reflections on the gaps in literature and what new areas of calculus research are needed.

Keywords Calculus · Trends in calculus research · Cycle of research and development · Future directions

Calculus plays an important role in secondary and tertiary education. Future teachers, engineers, doctors, economists, scientists, and, of course, mathematicians undertake the effort of learning and understanding calculus concepts and techniques. Calculus also carries status; taking a course in calculus is often thought to be a pinnacle of intellectual achievement by students and parents. Calculus is used as everything from a 'weeding out' course to fundamental preparation to take on applied problems in partner disciplines, preparing students to bring an understanding of rates, concavity, functional relationships, among other topics to bring to bear on multi-disciplinary problems. Large numbers of students enroll in calculus courses at the secondary and tertiary levels each year. For instance, in Europe and East Asia calculus is compulsory for many students at the secondary level, whereas in the United States, students take calculus either at the secondary or at the tertiary level (or both). Similarly, in most countries in Latin America, calculus is taken by millions of students at universities.

Although differences exist among countries, we argue that the secondary vs. tertiary differences are not great for a first course in differential and integral calculus, especially when taking a developmental lens. However, the secondary vs. tertiary differences might be greater when viewed through a pedagogical or cultural lens, including institutional constraints and affordances. This is an interesting and open area of research.

Given that a good number of students around the world enroll in calculus, whether at the secondary or tertiary level, research on the learning, teaching, and understanding of calculus has the potential to have broad impact. Thus, we argue it is fundamentally important that the body of research on calculus learning, teaching, and understanding coherently contribute to the practice of educating the millions of students who enroll in calculus courses each year.

Twenty years ago, Schoenfeld (1994) argued that well-executed research in collegiate mathematics education provides, "theoretically based, disciplined ways of enhancing our understanding of mathematical thinking, learning, and teaching" (p. 4). Has research in undergraduate mathematics education (and in calculus in particular)

C. Rasmussen (✉)
San Diego State University, San Diego, USA
e-mail: crasmussen@mail.sdsu.edu

K. Marrongelle
Portland State University, Portland, USA

M. C. Borba
São Paulo State University, Bela Vista, Rio Claro, SP, Brazil

produced results that enhance our understanding of mathematical thinking, learning, and teaching, as Schoenfeld argued it should? While the past several decades of research in calculus has contributed to better understanding of mathematical thinking, learning, and teaching in areas such as limit, derivative, and integral, too much research remains isolated and uncoordinated.

These gaps were the departure point of this issue of ZDM. With papers from over a dozen leaders in the field, this issue of ZDM takes partial stock of the current state of the field and exemplifies both the promise of research advances in calculus learning and teaching as well as the current limitations. In broad terms, the collection of papers shed light on the following questions: To what extent has the field moved forward in providing a more cohesive portrait of calculus teaching and learning? To what extent have research advances impacted the actual teaching of the millions of students that take calculus? In Sect. 2, we discuss the broad trajectories in which the field has advanced as well as a model for interpreting the cycle of research and development aimed at improving teaching and learning. In Sect. 3 we then use this background to discuss each of the papers in this volume.

1 Background

Research on calculus learning and teaching generally has followed a pattern of (1) identifying and studying student difficulties and cognitive obstacles followed by (2) investigations of the processes by which students learn particular concepts, (3) evolving into classroom studies (or close approximations thereof), including the effects of curricular and pedagogical innovations on student learning, and, more recently (4) research on teacher (including graduate student instructor, lecturers, etc.) knowledge, beliefs, and practices. We can see this pattern, in varying degrees, in the research in different subdomains of calculus: limit, derivative, and integral.

For example, in the subdomain of research on the learning and teaching of derivative, early research focused on students' difficulties and under-developed conceptual understandings of derivative (Orton 1983; Ferrini-Mundy and Graham 1991). An even earlier paper by Morgan and Warnock (1978) reported on an investigation of student difficulties as a result of calculating derivatives on a calculator. Similarly, early research on the learning and teaching of limit detailed a range of student difficulties and misconceptions (e.g., Davis and Vinner 1986; Furinghetti and Paola 1988; Tall and Vinner 1981).

In the 1980s and 1990s, advances in the theoretical foundations of student thinking and learning had a strong influence on research in mathematics education. The advent of constructivism in particular impacted undergraduate mathematics education research and led to the publication of research attempting to follow students through stages of understanding particular ideas. For example, the work of Dubinsky and colleagues adapted ideas from Piaget to develop what they refer to as the Action, Process, Object, Schema (APOS) theory (see Arnon et al. 2014 for a comprehensive review of APOS theory). At about the same time, Sfard (1991a, b) was developing related ideas of process and object and Gray and Tall (1994) were publishing on the idea of procepts. In physics education research, diSessa (1988) was fleshing out his theory of p-prims. While there are important differences between these different framings, the point we make is that these advances ushered in an era of research that was focused not on student misconceptions, but rather on articulating theories of how students learn and the role of manipulatives and other representations in learning (e.g., Cobb 1992; Thompson 1992). Zandieh (2000) illustrates the impact of these theoretical developments by combining a process-object layer with a context (representation) layer to form a framework for analyzing student understanding of the derivative. The research literature continues to investigate the role of representations in student learning. Hähkiöniemi (2004), for instance, reported on how students learn the concept of derivative through interactions with multiple representations.

In addition to building theories about how students learn, more recent research has also targeted the focus; for example, studying student understanding of the chain rule (Barbosa 2009; Kabael, 2010), the Riemann Sum (Sealey 2014), or the fundamental theorem of calculus (Salinas 2013; Thompson and Silverman 2008); bringing in ideas from psychology; for example, research on gesture (Yoon et al. 2011a, b), employing research related to psychonanalysis (Baldino and Cabral 1994), and crossing disciplinary boundaries to physics (Marrongelle 2004; Christensen and Thompson 2012). These are all important advances in our understanding of how students' learn specific calculus topics, but the studies leave the field with a hit-or-miss map of the terrain in calculus learning, teaching, and understanding.

Many of the studies in calculus learning, teaching, and understanding are relatively small in scale, implementing clinical interview methodologies with a small number students. Others, such as Borba and Villarreal (2005) and Soares (2012) have reported on studies that include whole classrooms (40–50 students) that investigate, for example, the role of visualization provided by different software in modeling or in problem solving activity.

The predominant methodology, whether applied to a few or many students, is the clinical interview. This begs the question, after several decades of research on student thinking about derivative, what do we know about the development of the derivative concept and how has the research shaped teaching practice? It is noteworthy that the research in calculus learning and teaching has not capitalized on advances in design research (Kelly et al. 2008) to further link theories of learning with theories of instructional design, as researchers in differential equations (Artigue 1994; Rasmussen 2007) and abstract algebra (Larsen et al. 2013) have. One might wonder why design research in calculus has not been more prevalent. One conjecture is that, at least in the US, the calculus reform movement in the 1990s was dominated by curriculum development projects led by thoughtful mathematicians who tended not to have extensive educational research expertise. The subsequent backlash from the calculus reform movement may have then had an effect on the type of calculus research that was carried out. We argue that the time is now right, given the depth of what we know about student learning of particular ideas in calculus, for the field to engage in comprehensive design research in which mathematicians and mathematics education researchers work together to address theoretical and pragmatic concerns related to the teaching and learning of calculus.

In its manuscript *Mathematical Proficiency for All Students: Toward a Strategic Research and Development Program in Mathematics Education*, the RAND Study Panel (2003) advised about the field of mathematics education writ large, "The absence of cumulative, well-developed knowledge about the practice of teaching mathematics and the limited links between research and practice have been major impediments to creating a system of school mathematics that works" (p. 5). The same observation can be made of calculus learning and teaching, *if* we value the interplay between the production of knowledge and the improvement of practice. The Cycle of Knowledge Production and Improvement of Practice (henceforth referred to simply as the Cycle of Knowledge) is one attempt to articulate a cycle of research and development aimed at improving learning and teaching (see Fig. 1). No single project, on its own, would be expected to answer a significant problem of practice or theory; rather, projects would build upon one another to build knowledge and inform practice.

If we again turn to the literature on thinking, learning and teaching of the derivative, we can ask how the derivative literature maps onto the Cycle of Knowledge. Many studies fall into the categories of "studies of basic problems of teaching and learning," including early research on student misconceptions (e.g., Orton 1983) and studies of students conceptions of derivative (e.g., Aspinwall et al.1997; Nemirovsky and Rubin 1992). Research informed by theories such as APOS (e.g., Asiala et al. 1997) moves to the "Development and testing of new theories of knowledge about teaching and learning" part of the cycle. As we continue around the cycle, we find that fewer published studies examine "Interventions in Practice" and even fewer report on teachers' mathematical knowledge for teaching derivative, an aspect of "Interventions in practice," a hugely under-investigated area in calculus learning and teaching research.

2 Papers in this issue

Including this introductory paper, there are 16 papers in this issue of ZDM dedicated to Arnold Kirsch. Kirsch's contributions to calculus research is acknowledged by including a reprint of one of his papers, which is characteristic for his interest in a deep and sound understanding of concepts beyond formalism. This reprint, which appears as

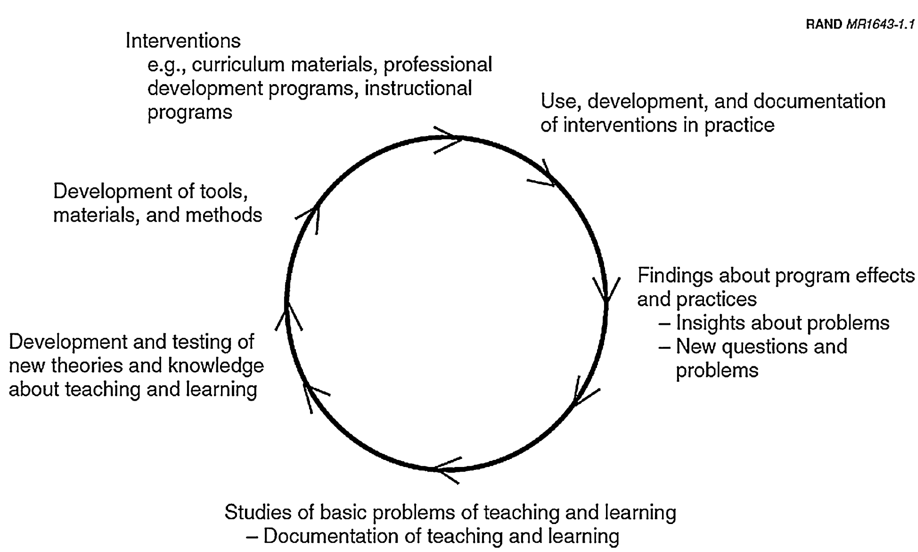

Fig. 1 Cycle of knowledge production and improvement of practice

the penultimate paper in this issue, is followed by a commentary paper by Werner Blum who remarks on the high importance of the work by Arnold Kirsch in the German and even wider European context. Preceding the reprint by Kirsch, there is a commentary paper by Barry Sloane, who provides an insightful perspective from the viewpoint of someone who is intimately familiar with a variety of research methods and who is keenly aware of the priorities and concerns of mathematics education research and its funding.

Thus, not counting this introductory paper, Kirsch's paper, and the two commentary papers, there are 12 other papers in this issue. These 12 papers, as well as the other 4 papers, reflect the growth of the research on calculus learning and teaching and exemplify both the progress made as well as the gaps still present. One noteworthy point is that none of the 12 papers is aptly characterized as primarily identifying and studying student difficulties and cognitive obstacles, which is more typical of educational research areas in their early stages. To be certain, all 12 papers discuss the challenges students and/or instructors typically face, but documenting such difficulties is not the main focus of the analysis. A more appropriate theme for all 12 papers is that of exploring the prospects and possibilities for more coherent and conceptual learning and teaching. More specifically, each of the 12 papers addresses one of the three research themes that typically follow reports that document student difficulties: investigating the processes by which students learn particular concepts; examining the effects of curricular and pedagogical innovations on student learning; and, Studying instructor knowledge, beliefs, and practices. In terms of the Cycle of Knowledge in Fig. 1, these three research themes represent movement clockwise starting at "studies of basic problems of teaching and learning". In the paragraphs that follow, we summarize the contributions of each of the 12 papers, starting with those that fit the research theme of "Investigating the processes by which students learn particular concepts", followed by the papers that primarily "examine the effects of curricular and pedagogical innovations on student learning", and conclude with the papers that "Study instructor knowledge, beliefs, and practices".

The theme of investigating the processes by which students learn particular concepts is embodied by only two of the 12 papers. In the paper by Swidan and Yerushalmy (2014, this issue), the authors make use of Radford's (2003) perspective on semiotics to detail the objectification processes involved in making sense of the concept of an indefinite integral when studied graphically in a dynamic technological environment. They analyze how 11 pairs of secondary school students think with and through cultural artifacts, which are taken to be essential sources of learning and shape thinking. Without significant intervention by a teacher or the researchers, but with the use of innovative digital tools, the pairs of students worked through a series of tasks and the corresponding analysis sheds light on the mathematical elements employed by the students and the paths that students followed as they explored dynamically linked graphs.

The paper by Kouropatov and Dreyfus (2014, this issue) takes a similar methodological approach of working with pairs of secondary school students outside of their regular classroom. They examine the work of four pairs students as they progress through a ten-lesson unit aimed at building the concept of definite integral. Making use of the theoretical perspective of abstraction in context (Hershkowitz et al. 2001), the authors detail how students can leverage the ideas of approximation and accumulation to develop a proceptual understanding of the integral in readiness for learning the fundamental theorem of calculus. In contrast to the paper by Swidan and Yerushalmy (2014, this issue), this research did not make use of digital technologies, in part because most secondary schools do not have access to specifically designed technologies and hence the authors wanted to mirror as much as possible typical school context.

The paper by Törner et al. (2014, this issue) represents a transition to studies that examine the effects of curricular and pedagogical innovations on student learning. In total, there are eight papers that fit this theme, four of which are empirical reports and four of which are more theoretically oriented reports. The Törner et al. paper reports on the intended secondary school calculus curriculum in several European countries. Through a careful literature review and small expert-based survey, these researchers found that, to a large extent, research advances (such as those in the previous two papers) that utilize digital technologies and/or specially designed tasks to actively engage students to develop conceptual learning of the more formal mathematics has not found its way into the majority of European secondary school calculus classrooms. A long-standing emphasis on procedures and techniques in conjunction with national exams with similar emphases is posited to contribute to the minimal extent to which research advances are finding their way into classrooms. An important contribution of the paper is in its careful comparison of secondary school calculus curriculum in several European countries, one of the first such comparisons.

While widespread adoption of innovative approaches to teaching and learning calculus has yet to be realized in several European countries (likely similar results exist around the world), some progress is being made with more local efforts. For example, the paper by Keene et al. (2014, this issue) report on an iterative, classroom-based design research study (Cobb et al. 2006) in a specialized calculus class for undergraduates who will become elementary school teachers. Their research program, which includes the development of a new curriculum, addresses a new department level requirement for prospective elementary school teachers to be conversant with the fundamental ideas in calculus. The study draws on the instructional design theory of Realistic Mathematics Education (Gravemeijer 1994) and the paper details the evolution of prospective elementary school teachers' intuitions and conceptions of the limit of a sequence. This work has potential to spread to other post secondary institutions in the United States that wish to increase the mathematical competencies required of prospective elementary school teachers, in part because it address the unique needs of future elementary school teachers, which are arguable different than that of students pursuing a degree in a science, technology, engineering, or mathematics.

Continuing the trend of exploring the prospects and possibilities for undergraduate students, but this time for students in the biological sciences, Soares and Borba (2014, this issue) report on a multi-level design research study (Lesh and Kelly 2000) that takes full advantage of digital technologies. Taking an epistemological perspective based on the notion of humans-with-media (Borba and Villarreal 2005), the authors focus on the role of the software during the evolution of one of the activities that developed the relation between secant lines and the instantaneous rate of change. Similar to the Keene et al. work, this research program has considerable potential for wider spread adoption for many of the same reasons. The needs of biology students are arguably different than those of mathematics majors.

As the field moves from studies that are conducted in settings outside of regular class time to design research studies that make use of regular classroom contexts, it is appropriate to carry out more quantitatively oriented studies that compare the effect of interventions in which the researchers have little to minimal involvement in the actual teaching. Such is the case with the paper by Code et al. (2014, this issue). Their work examines the efficacy of using research-based, high-engagement teaching methods to help undergraduate economics students master the conceptual and procedural aspects of calculus in a relatively standard differential calculus course taught in a large lecture format. The authors conducted a "switching replication" study, which makes for a strong quasi-experimental design (Shadish et al. 2001) as each student acts as their own control. Results include improved student performance—on conceptual items in particular—with a switching replication in that each section outperformed the other on the topic for which it received the intervention. We see this report as paradigmatic of a study that can be captured by more than one location of the Cycle of Knowledge. The original motivation is situated with studies of basic problems of teaching and learning, followed by the use, development, and documentation of interventions in practice.

Three other papers in this issue also address the effects of curricular and pedagogical innovations on student learning. In contrast to the previous empirical studies, these papers report on the theoretical and mathematical foundations for different approaches on how calculus ideas can (or should) be developed. In terms of the Cycle of Knowledge these four papers begin to address research focused on the "Development and testing of new theories and knowledge about teaching and learning" (see Fig. 1). For example, Weigand (2014, this issue) eschews the widespread embrace of starting with the limit to develop key ideas in calculus. He instead articulates a comprehensive discrete step-by-step approach by working with sequences and difference sequences, functions defined on Z, and discrete domains of Q to illustrate how one can develop the concept of rate of change. In this approach, digital technologies are used as a tool for the representation and visualization of sequences and functions as well as a tool to create recursively defined sequences, enabling the user to switch between symbolic, numerical, and graphical representations.

Rather than making use of a discrete approach, Moreno-Armella (2014, this issue) embraces the use of infinitesimals, such as conceiving of a smooth curve as a polygon whose sides have infinitesimal length. He argues that such an approach is especially valuable for teaching because it has a clear intuitive meaning for students. At the heart of Moreno-Armella's approach to calculus is the need for a clear distinction between analysis on the one hand, and the intuitive ideas of change and accumulation on the other. Moreno-Armella argues that the logical organization of analysis based on definitions of limit, real number, continuity, and so on, is not necessarily first from the cognitive viewpoint. Instead, through the use of dynamic digital technologies that shape and are shaped by the user, the Moreno-Armella tenders several examples of the possibilities for an approach to calculus that blends the intuitive ideas with the digital-dynamic embodiment of these ideas. This approach certainly has commonality with the stance Realistic Mathematics Education inspired approach taken by Keene et al. (2014, this issue).

For their part, Job and Schneider (2014, this issue) circumvent the dichotomy between formal and intuitive aspects of limits, which is an implicit or explicit concern of many of the papers in this issue. Using Chevallard's anthropological theory of the didactic (Chevallard 1999), they put forth an epistemological model where calculus is considered as a pragmatic praxeology that evolved into a deductive praxeology, the difference being in the type of tasks and the nature of the justifications. This allows them to distinguish between calculus, with its pragmatic praxeology, and analysis with its deductive praxeology. This distinction is similar to the one insisted on by Moreno-Armella. Developing their theoretical framing of the calculus to analysis transition further, they demonstrate how empirical positivism can function as an epistemological obstacle in both calculus and analysis. At the heart of the problem is that secondary schools and universities tend to blur the distinction between the two praxeologies, which reinforces the empirical positivist attitude as an epistemological obstacle to learning both calculus and analysis. Implications for both research and teaching are offered.

The final three papers in this issue represent more recent efforts in calculus research to study instructor knowledge, beliefs, and practices. For example, Eichler and Erens (2014, this issue) investigate the belief systems of 29 upper secondary school teachers, where belief systems refer to the intersection of cognitive and motivational aspects that influence the selection of content and goals for calculus teaching (Hannula 2012). The authors also compare teachers' belief systems to four education trends in the teaching of calculus, which provide a set of objectives from which calculus teachers potentially select their individual teaching goals. As the authors point out, research on calculus teachers' belief systems is scarce, and hence this report makes a useful contribution to an emerging area of calculus research.

The focus of the paper by Ellis et al. (2014, this issue) shifts from secondary school teacher belief systems (and the potential connection to instructional practices) to university level instructor practices and the relationship between these practices and student persistence in the calculus sequence. Based on regression analyses of data from a large national survey in the United States, these authors found that student persistence in calculus (a proxy for continuation in a STEM major) was related to different student reported frequencies of a number of pedagogical activities. Making use of Tinto's (2004) framework on persistence, the authors discuss a number of implications of this work for retaining students in a STEM major. In the United States and elsewhere around the world, there is great need to better understand the factors that contribute to student decisions to stay in or to leave a STEM major, and this paper makes a useful contribution in this direction.

Finally, the paper by White and Mesa (2014, this issue) examines a different aspect of instructor practice, namely the potential cognitive demand of nearly 5,000 Calculus I tasks that instructors assigned to students at a 2-year college in the United States identified as having a more successful calculus program. Certainly the tasks that students are assigned are essential learning opportunities, and hence understanding the nature of these tasks and how assigned tasks may vary across instructors is an important aspect of instructional decision-making. One of their findings is that nearly half of tasks in the data corpus were "complex procedures" or "rich tasks". This is especially interesting in light of the result from the paper by Törner et al. (2014, this issue), who found that European secondary school students were primarily being exposed to procedures and techniques. Another interesting finding by White and Mesa is the fact that there was considerable variation in the nature of the tasks assigned across the five instructors at the institution studied, despite the fact that they all used the same textbook. The authors discuss a number of implications of this finding for tertiary institutions. Finally, the framework they developed for charactering tasks is an important contribution in and of itself. It was shown to be reliable in coding tasks and extended relevant prior work and hence will likely be a valuable tool for other researchers who wish to examine the opportunities to learn that different tasks afford students.

3 Conclusion

One salient characteristic of this issue of ZDM is that it reports on research on early topics of calculus, and this focus corroborates the findings of Britton and Henderson (2013), who reviewed all the proceedings of the Delta conferences. They point out that there a shortage of papers in the Delta Conference proceedings that go beyond early topics of calculus, in areas such as multivariable calculus and differential equations. A look at PME proceedings will confirm this finding.

As the papers in this volume attest to, the research in calculus is continuing to advance our foundational knowledge of the learning and teaching process. While we know much about how students learn particular ideas in calculus and the potential for digital technologies and high engagement pedagogies, there still exists the issue of how researchers can coordinate various advances that are grounded in and informed by different theoretical perspectives. Consider the two papers in this volume on the student learning of the indefinite and definite integral (Kouropatov and Dreyfus 2014, this issue; Swidan and Yerushalmy 2014, this issue). On its own, each paper makes progress on revealing the processes by which students can learn the indefinite and definite integral, but each paper views the process of learning in fundamentally different ways. How then, should the research field coordinate such advances? To be sure, the research field is currently working, both theoretically and pragmatically, on ways to coordinate and network different theoretical perspectives and respective findings (e.g., Artigue and Mariotti 2014; Bikner-Ahsbahs and Prediger 2014; Hershkowitz et al. 2014; Prediger et al. 2008; Rasmussen et al. 2012), but considerably more progress is needed.

In light of the challenges that the research community has in coordinating different research advances, it is perhaps not surprising that these advances have not had a widespread impact in the actual teaching of and learning of calculus (e.g., Toerner et al. this issue). The relationship between the progress researchers need to make to coordinate and network different advances and the uptake of such advances by the broader practitioner and policymaking communities is an issue that sorely needs addressing. Moreover, taking up research advances in any widespread manner requires an understanding of learners, teachers, classrooms, departments, and institutions as complex systems. While some local improvements and innovations are occurring in individual classrooms, broader impact will need to based on theoretical advances and empirical studies that advance what we know about how institutions in all their complexity change. That is, research that takes up the institutional and cultural context and how these aspects constrain and enable sustained uptake of advances in calculus learning and teaching is sorely needed. Indeed, we argue that this represents a new research theme, one yet to be realized to any large extent.

Lastly, the papers in this issue reveal another tension in the research on calculus—what exactly do we want students in calculus to learn? The tension here is multifaceted and includes differences between the secondary and tertiary level as well as differences between the needs of students studying mathematics and those majoring in economics, biology, engineering, business, etc. In a deeply theoretical way with considerable practical implications, Job and Schneider (2014, this issue) reveal this tension as it often plays out between secondary school calculus and university calculus. The distinction between pragmatic and deductive praxeologies is widespread and likely many involved in the teaching of calculus have yet to appreciate the depth of the gulf that separates the two praxeologies. At the heart of the issue is how researchers, curriculum developers, practitioners, and policy makers think about the relationship between calculus, with its more intuitive foundation of how fast, how much, and infinitesimal thinking on the one hand, and analysis, with its formal treatment of limits and continuity, on the other. Moreover, how the various constituents think about this issue should depend on the questions for whom and when? To what extent do students in the biological sciences (or engineering or physics for that matter) benefit from learning about the formal role of limits? When do mathematics majors benefit most from the formal treatment of limits? What might be the role of modeling in calculus for the learning of calculus itself and in preparation for subsequent study of differential equations, fluid dynamics, electromagnetism, and so on?

Progress on mathematics courses after differential and integral calculus is underway. For, example, focusing on the transition from Calculus to Analysis, Alves (2012) investigates the use of Geogebra to help students visualize the relation of epsilon and N in understanding convergence of series and Bergé (2008) examines the opportunities for learning about the set of real numbers in four undergraduate correlative courses in Calculus and Analysis. Trigueros and Martínez-Planell (2010) examine student learning of two variable functions in multivariable calculus. The research of Dullius, Araujo, and Veit (2011), Javaroni (2007), Keene (2007), and Soares (2012) aim to improve the teaching learning process of differential equations and explore the potential of computers to promote favorable conditions for meaningful learning. These recent research efforts are only but a few of the international contributions that are providing much needed insights into the learning and teaching of mathematical concepts that build on and extend the research on early calculus. An area of even greater need, however, is that of the relationship between calculus and the client disciplines of engineering, physics, biology, and chemistry. We end this paper with a call for research that closely examines the ways in which calculus ideas are leveraged in the client disciplines, how these ideas are conceptualized and represented in the client disciplines, and what these insights might mean for calculus instruction.

References

Alves, F. R. V. (2012). Exploração de noções topológicas na transição do Cálculo para a Análise Real com o GeoGebra. *Revista Do Instituto Geogebra Internacional de São Paulo, 1*, 165–179.

Arnon, I., Cottrill, J., Dubinsky, E., Oktac, A., Fuentes, S., Trigueros, M., et al. (2014). *APOS theory: A framework for research and curriculum development in mathematics education*. New York, NY: Springer.

Artigue, M. (1994). Didactical engineering as a framework for the conception of teaching products. In R. Biehler, R. W. Scholz, R. Sträßer, & B. Winkelmann (Eds.), *Didactics of mathematics as a scientific discipline* (pp. 27–39). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Artigue, M., & Mariotti, M. A. (2014). Networking theoretical frames: the ReMath enterprise. *Educational Studies in Mathematics, 85*, 329–355.

Asiala, M., Cottrill, J., Dubinsky, E., & Schwingendorf, K. E. (1997). Networking theoretical frames: the development of students' graphical understanding of the derivative. *The Journal of Mathematical Behavior, 16*(4), 399–431.

Aspinwall, L., Shaw, K. L., & Presmeg, N. C. (1997). Uncontrollable mental imagery: graphical connections between a function and its derivative. *Educational Studies in Mathematics, 33*(3), 301–317.

Baldino, R. R., & Cabral, T. C. B. (1994). Os quatro discursos de Lacan eo Teorema Fundamental do Cálculo. Revista Quadrante, pp. 1–24.

Barbosa, S.M. (2009). *Tecnologias da Informação e Comunicação, Função Composta e Regra da Cadeia*. Doctoral Dissertation, UNESP, Rio Claro, Brazil.

Bergé, A. (2008). The completeness property of the set of real numbers in the transition from calculus to analysis. *Educational Studies in Mathematics, 67*, 217–235.

Bikner-Ahsbahs, A., & Prediger, S. (2014) (Eds.), *Networking of theories as a research practice*. New York, NY: Springer.

Borba, M. C., & Villarreal, M. E. (2005). *Humans-with-media and the reorganization of mathematical thinking: Information and communication technologies, modeling, experimentation and visualization*. New York, NY: Springer.

Britton, S., & Henderson, J. (2013) Issues and trends: a review of Delta conference papers from 1997 to 2011. *Lighthouse Delta 2013: The 9th Delta Conference on teaching and learning of undergraduate mathematics and statistics*, pp. 50–58.

Chevallard, Y. (1999). L'analyse des pratiques enseignantes en théorie anthropologique du didactique. *Recherches en Didactique des Mathématiques, 19*(2), 221–266.

Christensen, W. M., & Thompson, J. R. (2012). Investigating graphical representations of slope and derivative without a physics context. *Physical Review Special Topics-Physics Education Research*, *8*(2). http://journals.aps.org/prstper/abstract/10.1103/PhysRevSTPER.8.023101.

Cobb, P. (1992). A constructivist alternative to the representational view of mind in mathematics education. *Journal for Research in Mathematics Education, 23*(1), 2–33.

Cobb, P., Confrey, J., diSessa, A., Lehrer, R., & Schaubel, L. (2006). Design experiments in educational research. *Educational Researcher, 32*(1), 99–113.

Code, W., Piccolo, C., Kohler, D., & MacLean, M. (2014, this issue). Teaching methods comparison in a large calculus class. *ZDM—The International Journal on Mathematics Education*. doi:10.1007/s11858-014-0582-2.

Davis, R. B., & Vinner, S. (1986). The notion of limit: some seemingly unavoidable misconception stages. *Journal of Mathematical Behavior, 5*(3), 281–303.

diSessa, A. A. (1988). Knowledge in pieces. In G. Forman & P. Pufall (Eds.), *Constructivism in the computer age* (pp. 49–70). Hillsdale, NJ: Lawrence Erlbaum Associates.

Dullius, M. M., Araujo, I. S., & Veit, E. A. (2011). Ensino e Aprendizagem de Equações Diferenciais com Abordagem Gráfica, Numérica e Analítica: um experiência em Cursos de Engenharia. Bolema. *Boletim de Educação Matemática* (UNESP. Rio Claro. Impresso), *Vol. 24*, pp. 17–42.

Eichler, A., & Erens, R. (2014, this issue). Teachers' beliefs towards teaching calculus. *ZDM—The International Journal on Mathematics Education*. doi:10.1007/s11858-014-0606-y.

Ellis, J., Kelton, M. L., & Rasmussen, C. (2014, this issue). Student perceptions of pedagogy and associated persistence in calculus. *ZDM—The International Journal on Mathematics Education*. doi:10.1007/s11858-014-0577-z.

Ferrini-Mundy, J., & Graham, K. (1991). An overview of the calculus curriculum reform effort: issues for learning, teaching, and curriculum development. *American Mathematical Monthly, 98*(7), 627–635.

Furinghetti, F., & Paola, D. (1988). Wrong beliefs and misunderstandings about basic concepts of calculus (age 16–19). In *Proceedings of the 39th rencontre internationale de la CIEAEM, 1987* (pp. 173-177). Sherbrooke, Canada.

Gravemeijer, K. (1994). *Developing realistic mathematics education*. Utrecht, The Netherlands: CD Press.

Gray, E., & Tall, D. (1994). Duality, ambiguity, and flexibility: a "proceptual" view of simple arithmetic. *Journal for Research in Mathematics Education, 25*(2), 116–140.

Hähkiöniemi, M. (2004). Perceptual and symbolic representations as a starting point of the acquisition of the derivative. In Hoines, M. J. & Fuglestad, A. B. (Eds.), *Proceedings of the 28th Annual Meeting of the International Group for the Psychology of Mathematics Education*, *Vol. 3*, pp. 73–80.

Hannula, M. S. (2012). Exploring new dimensions of mathematics-related affect: embodied and social theories. *Research in Mathematics Education, 14*(2), 137–161.

Hershkowitz, R., Schwarz, B. B., & Dreyfus, T. (2001). Abstraction in context: epistemic actions. *Journal for Research in Mathematics Education, 32*, 195–222.

Hershkowitz, R., Tabach, M., Rasmussen, C., & Dreyfus, T. (2014). Knowledge shifts in a probability class: a case study. *ZDM—The International Journal on Mathematics Education*. doi:10.1007/s11858-014-0576-0.

Javaroni, S. (2007). *Abordagem geométrica: possibilidades para o ensino e aprendizagem de Introdução às Equações Diferenciais Ordinária, Doctoral Disssertation*. Rio Claro, Brazil: UNESP.

Job, P., & Schneider, M. (2014, this issue). Empirical positivism, an epistemological obstacle in the learning of calculus. *ZDM—The International Journal on Mathematics Education*. doi:10.1007/s11858-014-0604-0.

Kabael, T. (2010). Cognitive development of applying the chain rule through three worlds of mathematics. *Australian Senior Mathematics Journal, 24*(2), 14–28.

Keene, K. (2007). A characterization of dynamic reasoning: reasoning with time as parameter. *Journal of Mathematical Behavior, 26*, 230–246.

Keene, K. A., Hall, W., & Duca, A. (2014, this issue). Sequence limits in calculus: using design research and building on intuition to support instruction. *ZDM—The International Journal on Mathematics Education*. doi:10.1007/s11858-014-0597-8.

Kelly, A. E., Lesh, R. A., & Baek, J. Y. (Eds.). (2008). *Handbook of design research methods in education: Innovations in science, technology, engineering, and mathematics learning and teaching*. London: Routledge.

Kouropatov, A., & Dreyfus, T. (2014, this issue). Learning the integral concept by constructing knowledge about accumulation. *ZDM—The International Journal on Mathematics Education*. doi:10.1007/s11858-014-0571-5.

Larsen, S., Johnson, E., & Weber, K. (2013). (Eds.) The teaching abstract algebra for understanding project: designing and scaling up a curriculum innovation. *Journal of Mathematical Behavior, 32*(4).

Lesh, R., & Kelly, A. (2000). Multi-tiered teaching experiments. In A. Kelly & R. Lesh (Eds.), *Handbook of research in mathematics and science education* (pp. 197–230). Mahwah, NJ: Lawrence Erlbaum.

Marrongelle, K. (2004). Context, examples, and language: students uses of physics to reason about calculus. *School Science and Mathematics, 10*(6), 258–272.

Moreno-Armella, L. (2014, this issue). An essential tension in mathematics education. *ZDM—The International Journal on Mathematics Education*. doi:10.1007/s11858-014-0580-4.

Morgan, R. V., & Warnock, T. T. (1978). Derivatives on the handheld calculator. *Mathematics Teacher, 71*(6), 532–537.

Nemirovsky, R., & Rubin, A. (1992). *Students' tendency to assume resemblances between a function and its derivative*, TERC Working Paper 2–92, Cambridge.

Orton, A. A. (1983). Students' understanding of differentiation. *Educational Studies in Mathematics, 14*(3), 235–250.

Prediger, S., Bikner-Ahsbahs, A., & Arzarello, F. (2008). Networking strategies and methods for connecting theoretical approaches: first steps towards a conceptual framework. *ZDM—The International Journal on Mathematics Education, 40*(2), 165–178.

Radford, L. (2003). Gestures, speech, and the sprouting of signs: a semiotic-cultural approach to students' types of generalization. *Mathematical Thinking and Learning, 5*(1), 37–70.

RAND Mathematics Study Panel. (2003). *Mathematical proficiency for all students: Toward a strategic research program in mathematics education*. Santa Monica, CA: RAND Corporation.

Rasmussen, C. (Ed.) (2007). An inquiry oriented approach to differential equations. *Journal of Mathematical Behavior, 26*(3) (special issue).

Rasmussen, C., Wawro, M., & Zandieh, M. (2012). *Four lenses for examining individual and collective level mathematical progress.* Paper presented at the annual meeting of the American Educational Research Association, Vancouver, Canada.

Salinas, P. (2013). Approaching calculus with SimCalc: linking derivative and antiderivative. In S. J. Hegedus & J. Roschelle (Eds.), *The SimCalc vision and contributions* (pp. 383–399). Dordrecht, The Netherlands: Springer.

Schoenfeld, A. (1994). Some notes on the enterprise (research in collegiate mathematics education, that is). *Research in Collegiate Mathematics Education, 1*, 1–19.

Sealey, V. (2014). A framework for characterizing student understanding of Riemann sums and definite integrals. *The Journal of Mathematical Behavior, 33*, 230–245.

Sfard, A. (1991a). On the dual nature of mathematical conceptions: reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics, 22*, 1–36.

Sfard, A. (1991b). On the dual nature of mathematical conceptions: reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics, 22*(1), 1–36.

Shadish, W. R., Cook, T. D., & Campbell, D. T. (2001). *Experimental and quasi-experimental designs for generalized causal inference*. Boston: Houghton Mifflin.

Soares, D. S. (2012). Uma abordagem pedagógica baseada na análise de modelos para alunos de biologia: qual o papel do software? Doctoral Dissertation, UNESP, Brazil.

Soares, D. S., & Borba, M. (2014, this issue). The role of software Modellus in a teaching approach based on model analysis. *ZDM—The International Journal on Mathematics Education*. doi:10.1007/s11858-013-0568-5.

Swidan, O., & Yerushalmy, M. (2014, this issue). Learning the indefinite integral in a dynamic and interactive technological environment. *ZDM—The International Journal on Mathematics Education*. doi:10.1007/s11858-014-0583-1.

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics, 12*(2), 151–169.

Thompson, P. W. (1992). Notations, conventions, and constraints: contributions to effective uses of concrete materials in elementary mathematics. *Journal for Research in Mathematics Education, 23*, 123–147.

Thompson, P. W., & Silverman, J. (2008). The concept of accumulation in calculus. In M. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics* (pp. 117–131). Washington, DC: The Mathematical Association of America.

Tinto, V. (2004). Linking learning and leaving. In J. M. Braxton (Ed.), *Reworking the student departure puzzle*. Nashville, TN: Vanderbilt University Press.

Törner, G., Potari, D., & Zachariades, T. (2014, this issue). Calculus in European classrooms: curriculum and teaching in different educational and cultural contexts. *ZDM—The International Journal on Mathematics Education*. doi:10.1007/s11858-014-0612-0.

Trigueros, M., & Martínez-Planell, R. (2010). Geometrical representations in the learning of two-variable functions. *Educational Studies in Mathematics, 73*(1), 3–19.

Weigand, H.-G. (2014, this issue). A discrete approach to the concept of derivative. *ZDM—The International Journal on Mathematics Education*. doi:10.1007/s11858-014-0595-x.

White, N., & Mesa, V. (2014, this issue). Describing cognitive orientation of Calculus I tasks across different types of coursework. *ZDM—The International Journal on Mathematics Education*. doi:10.1007/s11858-014-0588-9.

Yoon, C., Thomas, M. J., & Dreyfus, T. (2011a). Grounded blends and mathematical gesture spaces: developing mathematical understandings via gestures. *Educational Studies in Mathematics, 78*(3), 371–393.

Yoon, C., Thomas, M. J., & Dreyfus, T. (2011b). Gestures and insight in advanced mathematical thinking. *International Journal of Mathematical Education in Science and Technology, 42*(7), 891–901.

Zandieh, M. (2000). A theoretical framework for analyzing student understanding of the concept of derivative. *Research in Collegiate Mathematics Education*. *IV*, 103–127.